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Analytic representations for circle-jump moments

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Abstract. Herein we study s-th moments $\langle|\vec{R}_n|^s\rangle$ for a particular "circle-jump" coordinate $\vec{R}_n$ in the plane. Recently, Borwein et al. [3] have studied a random-walk integral that coincides precisely with said s-th moment. Using sophisticated algebra and combinatorics, those researchers have developed some fascinating relations and conjectures for such integrals. Herein we use physical, probabilistic notions to provide exact analytic representations for the moments, for any complex s and $n = 1, 2, 3, 4$. These analytic formulae support well the Borwein et al. conjectures. There is a theoretical blockade at $n = 5$, as has happened in recent years for similar multidimensional integration scenarios. A byproduct of the present analysis is a collection of low-dimensional integrals that may well allow precise quadrature for large dimension n.

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1 Random-walk integrals as expectations

The random-walk integral of Borwein et al. [3] is

$$W_n(s) \ := \ \int_{\vec{r} \in [0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi i r_k} \right|^s \ \mathcal{D}\vec{r} \tag{1}$$

where by $\mathcal{D}\vec{r} := dr_1 dr_2 \cdots dr_n$ we mean the n-dimensional volume element. We note right off that $W_n(s)$ is a generalized box integral, in the sense that there is a specific integrand with the domain of integration being the unit box $[0,1]^n$. Previously defined "box integrals" in the literature are the specific cases where the integrand is $|\vec{r}|^s$, as in [2].

So, while previous box integrals are actually expectations $\langle|\vec{r}|^s\rangle$ over the unit n-box, it is evident that $W_n(s)$ is the expectation $\langle|\vec{R}_n|^s\rangle$ where $\vec{R}_n$ is defined as the position of a "circle-jump" random walk after n steps. To specify the underlying random-walk model, then: A particle starts at the origin, jumps a unit distance in a random direction, then jumps again a unit distance with random direction, and so on. It is easily seen that the absolute-valued exponential sum in (1) is precisely the distance $|\vec{R}_n|$; thus W_n is an expectation upon the n-th jump.[1]

We define the radial density after n steps as (here and elsewhere, we denote $r := |\vec{r}|$)

$$\phi_n(r)\ dr \ := \ \text{Prob}(\text{particle } \in (r, r+dr)).$$

Note that this is *not* the same as the spatial density; the latter is

$$\Phi_n(\vec{r}) \ := \ \frac{\phi_n(r)}{2\pi r},$$

so that we have automatic normalizations

$$\int_0^\infty \phi_n(r)\ dr \ = \ 1,$$

and an integral over the plane,

$$\int_{\vec{r} \in \mathcal{R}^2} \Phi_n(\vec{r})\ \mathcal{D}\vec{r} \ = \ 1,$$

where we are using the polar-coordinate area element $\mathcal{D}\vec{r} = r\ dr\ d\theta$.

Referring to Figures 1 and 2, we can see some attractive—perhaps somewhat nonintuitive—displays of jump densities for certain W_n.

[1] The present author is aware that this circle-jump walk is of course the classical underpinning of the integral (1) itself. The point being, one can sometimes infer just from a given integral form an appropriate statistical model.

It arises immediately from the circle-jump model that we have precisely

$$W_n(s) \;=\; \int_0^\infty r^s \phi_n(r)\; dr,$$

and in fact, the upper limit on this integral can be taken to be $n+\epsilon$, say $n+1$ if convenient, because *the density must vanish* for $r > n$ (the walk after n steps cannot be farther than n away from the origin).

2 Initial probabilistic inferences

Moving along with the notion that $W_n(s)$ is an expectation $\langle |R_n|^s \rangle$, we know trivially that

$$W_n(0) = 1,$$

and, from classical probability principles we know that

$$W_n(2) = n,$$

because the variance after n steps is n times the 1-step variance, which itself is 1.

Next, the classical central-limit theorem tells us that, since we know that the n-jump variance is just n, the radial density for large n behaves as

$$\phi_n(r) \quad \sim_{n\to\infty} \quad \frac{2r}{n} e^{-r^2/n}.$$

This leads to the immediate estimate

$$W_n(s) \quad \sim_{n\to\infty} \quad n^{s/2}\, \Gamma(1 + s/2).$$

It is not immediately clear how one should interpret this asymptotic estimate physically, say, but as a leading asymptotic term it is useful for checking other results.

A particularly attractive—and useful—observation of Borwein et al. [3] is the dimensional-reduction formula

$$W_n \;=\; \int_{\vec{r}\in[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi i r_k} \right|^s \; \mathcal{D}\vec{r}.$$

Not only does the circle-jump model establish this, but the integrand may be replaced with

$$\left| e^{2\pi i \phi} + \sum_{k=1}^{n-1} e^{2\pi i r_k} \right|^s,$$

where ϕ is *any real number*. The proof is plain and simple: One may imagine the very first jump to be at fixed angle ϕ without changing the expectation of $|\vec{R}_n|$. Because of the freedom in choosing ϕ, we may immediately derive some moment results. For example, choosing $\phi = 0, \pi$ we have, for nonnegative integers K:

$$2W_n(2K) = \int_{\vec{r}\in[0,1]^{n-1}} \left(\left| 1 + \sum_{k=1}^{n-1} e^{2\pi i r_k} \right|^{2K} + \left| -1 + \sum_{k=1}^{n-1} e^{2\pi i r_k} \right|^{2K} \right) \mathcal{D}\vec{r},$$

in which the ± 1 choices in front of each sum allow for convenient cancellation. It is especially simple to derive $W_n(0) = 1, W_n(2) = n$ from this integral. One may press the integration further to derive the beautiful relation of Borwein et al. [3] for nonnegative integers K:

$$W_n(2K) \;=\; \sum_{a_1+\cdots+a_n=K} \binom{K}{a_1, \ldots, a_n}^2.$$

We may recast the combinatorics in several ways, for example summing over a_n we obtain

$$W_n(2K) = \sum_{a=0}^{K} \binom{K}{a}^2 W_{n-1}(2a).$$

This recursion can be used to show that *the random-walk integral* $W_n(2K)$ *is a polynomial in* n *of degree* K. Moreover, the leading coefficient (of n^K) is $K!$; this being entirely consistent with our previous central-limit asymptotics.

Thus for example,

$$W_n(0) = 1,$$
$$W_n(2) = n,$$
$$W_n(4) = 2n^2 - n,$$
$$W_n(6) = 6n^3 - 9n^2 + 4n.$$

It would be good to have a quick proof of the polynomial property; or better, closed-form coefficients. In any case, such polynomials are valuable in testing new representations and conjectures regarding $W_n(s)$.

3 Deeper probabilistic analysis

3.1 Closed-form convolutions

Convolution is a powerful expedient for general random-walk analysis. Even some highly nonstandard walks can be analyzed via convolution; as just one example, the present author found such methods invaluable for the treatment of "running-out-of-fuel" walks [5]. In fact, there is an historical connection between the jump-walks and convolution, which connection going back a full century [3], [6]. One might consider the present paper section as a bridge between convolution theory and symbolic analysis.

For the $n = 1$ circle jump, the radial density function is

$$\phi_1(r) \; = \; \delta(r-1),$$

whose associated spatial density function is

$$\Phi_1\left(\vec{r}\right) \; = \; \frac{1}{2\pi r}\delta(r-1),$$

where as usual $r := |\vec{r}|$. (In this Φ_1 formula, one may of course omit the r in the denominator, due to the delta function presence.) The $n = 2$ density functions can be obtained via convolution in the following way

$$\Phi_2\left(\vec{r}\right) \; = \; \int_{\mathcal{R}^2} \Phi_1\left(\vec{r}-\vec{q}\right)\Phi_1\left(\vec{q}\right)\; \mathcal{D}\vec{q}$$

$$= \; \int_{\mathcal{R}^2} \frac{1}{2\pi|\vec{r}-\vec{q}|}\,\delta(|\vec{r}-\vec{q}|-1)\,\frac{1}{2\pi q}\,\delta(q-1)\;\mathcal{D}\vec{q}.$$

Now we can employ a technique often found in theoretical physics treatments; namely, if a function $h(z)$ has a simple zero at $z = z_0$, then we have, formally speaking

$$\delta(h(z)) \; = \; \frac{\delta(z-z_0)}{|h'(z_0)|},$$

which substitution performed at every simple zero in the (open) integration domain. After careful manipulation of the previous integral we arrive at the closed form for the spatial density after $n = 2$ jumps, namely

$$\Phi_2(\vec{r}) \; = \; \frac{1}{\pi}\,\Re : \frac{1}{r}\frac{1}{\sqrt{1-r^2/4}}.$$

Here, the real-value extraction is just a convenience to force vanishing of Φ_2 for $r > 2$. Note the agreement between this exact form for Φ_2 and the lower-left ($n = 2$) plot of Figure 1. (In the mathematical reality, both the central singularity and the ring singularity diverge to infinity, so the figure is a qualitative pictorial as it were.)

Even at $n = 3$ this exact-convolution procedure becomes stultifying. We have

$$\Phi_3\left(\vec{r}\right) \; = \; \int_{\mathcal{R}^2} \Phi_2\left(\vec{r}-\vec{q}\right)\Phi_1\left(\vec{q}\right)\; \mathcal{D}\vec{q}$$

$$= \; \int_{\mathcal{R}^2} \Phi_2\left(\vec{r}-\vec{q}\right)\,\frac{1}{2\pi q}\,\delta(q-1)\;\mathcal{D}\vec{q}.$$

$$= \frac{1}{\pi^3} \Re : \int_0^2 \frac{1}{\sqrt{1-\frac{q^2}{4}}\sqrt{q^2-(r-1)^2}\sqrt{(r+1)^2-q^2}} \, dq.$$

The present author does not know a closed form for this integral.[2] Note that any desired walk integrals $W_3(s)$ can be in principle calculated as $2\pi \int_0^3 \Phi_3(\vec{r}) r^{s+1} \, dr$.

3.2 Convolution theorem

Since we have reached an impasse in regard to direct convolution, we choose now to change direction by employing the powerful convolution theorem.

Being as the characteristic spectral function of the $n = 1$ circle jump is $J_0(kr)$, meaning that

$$\Phi_1(\vec{r}) \;=\; \frac{1}{2\pi r}\delta(r-1) \;=\; \frac{1}{2\pi} \int_{\mathcal{R}^2} J_0(kr) e^{i\vec{k}\cdot\vec{r}} \, \mathcal{D}\vec{k},$$

the convolution theorem relating spatial convolution to spectral products gives immediately the n-general case

$$\Phi_n(\vec{r}) \;=\; \frac{1}{2\pi} \int_{\mathcal{R}^2} J_0(kr)^n e^{i\vec{k}\cdot\vec{r}} \, \mathcal{D}\vec{k}.$$

From this n-general case we find a *one*-dimensional representation for the radial density, as

$$\phi_n(r) \;=\; r \int_0^\infty k \; J_0^n(k) \; J_0(kr) \; dk,$$

Being as

$$\int_0^\infty x^\rho J_0(x) \; dx \;=\; \frac{2^\rho \Gamma\left(\frac{\rho+1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\rho}{2}\right)},$$

we have, formally and without immediate regard to convergence issues, a general representation

$$W_n(s) \;=\; \int_0^\infty r^s \phi_n(r) \; dr \;=\; s \, 2^s \frac{\Gamma(s/2)}{\Gamma(-s/2)} \int_0^\infty k^{-s-1} \; J_0^n(k) \; dk. \quad (2)$$

It is important to observe at this juncture that, since $\phi_n(r)$ vanishes for $r > n$, we also have a 2-dimensional integral

$$W_n(s) \;= \int_0^{n+1} dr \int_0^\infty dk \;\, J_0(k)^n \;\, J_0(kr) \, k \, r^{s+1} \quad (3)$$

[2] However, numerical integration appears to agree with the top plot in Figure 2; said figure having been computed using a separate quadrature method described later, so there is some consistency here, even in the absence of rigorous knowledge.

which may be suitable for numerical quadrature. This can be put in an equivalent form, now a 1-dimensional integral

$$W_n(s) = \frac{(2\,n)^{s+2}}{s+2} \int_0^\infty {}_1F_2\left(\frac{s}{2}+1; 1, \frac{s}{2}+2; -n^2x^2\right)\, x\, J_0\left(0,\, x\right)^n\, dx. \tag{4}$$

However, it is not yet clear whether the previous, 2-dimensional integral over k, r is experimentally inferior to this hypergeometric instance.

One observation that might be important for Bessel quadrature in, say, (3): In the treatment [4] there are so-called "exp-arc" expansions for $J_\mu(z)$, integer μ. These are *not* asymptotic expansions, and so may be used with impunity to compute tails of Bessel integrals.

4 Analytic moment representations for $n = 1, 2, 3, 4$

Representation (2) for $W_n(s)$ involves an integral that converges on a strip of finite width in the complex s-plane. But this is enough to infer analytic continuation of the evaluated integral. To this end, we use various moment integrals

$$\mu_n(s) \;:=\; \int_0^\infty k^{-s-1} J_0(k)^n\; dk,$$

which are known in closed analytic form for $n = 1, 2, 3, 4$ at least. Note the connection with previous research into Bessel-moment integrals (see [1] and references therein, particularly the work of V. Adamchik).

From (2) we have

$$W_n(s) = \; s\, 2^s \frac{\Gamma(s/2)}{\Gamma(-s/2)} \mu_n(s),$$

so that whenever we know a Bessel moment, we know W_n. The results of symbolic computation and analytic inference are the closed forms[3]

$$W_1(s) \;=\; 1,$$

$$W_2(s) \;=\; \frac{2^{s-1} s \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{s}{2}+1\right)^2} \;=\; \binom{s}{s/2},$$

with these $n = 1, 2$ cases already resolved in superb combinatorial fashion by Borwein et

[3]A decent symbolic processor of today asked for Bessel moments should yield the forms we display, and do so rather quickly.

al. [3]. Moving along, we have

$$W_3(s) \;=\; \frac{s\Gamma\left(-\frac{s}{2}-\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{4\pi\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{s}{2}+\frac{3}{2},\frac{s}{2}+\frac{3}{2};\frac{1}{4}\right)\;+ \tag{5}$$
$$\frac{2^s\Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{s}{2}+1\right)}\,{}_3F_2\left(-\frac{s}{2},-\frac{s}{2},-\frac{s}{2};1,\frac{1}{2}-\frac{s}{2};\frac{1}{4}\right).$$

It is of interest that this exact W_3 representation has evident, yet specious singularities. For example, at $s = 1$ the two hypergeometric summands here (including their respective Γ-function prefixes) *each* diverge, yet cancellation occurs, as in

$$W_3(1+10^{-30}) \approx 1.574597237551893657494692183\,0,$$

which agrees with Borwein et al. [3] numerically.

Another hypergeometrically resolvable dimension is $n = 4$:

$$W_4(s) \;=\; \frac{2^s\Gamma\left(-\frac{s}{2}-\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+1\right)^2}{\pi^{3/2}\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{s+3}{2}\right)^2}\,{}_4F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{s}{2}+1;\frac{s}{2}+\frac{3}{2},\frac{s}{2}+\frac{3}{2},\frac{s}{2}+\frac{3}{2};1\right)\;+ \tag{6}$$
$$\frac{2^s\Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{s}{2}+1\right)}\,{}_4F_3\left(\frac{1}{2},-\frac{s}{2},-\frac{s}{2},-\frac{s}{2};1,1,\frac{1}{2}-\frac{s}{2};1\right).$$

Again there are specious-singularity issues, yet limits can be taken; for example, we calculate from (6):

$$W_4(1+10^{-15}) \approx 1.799092479842851,$$

again in agreement with the numerics of Borwein et al. [3].

4.1 Singularity removal

One might ask whether the pole cancellation in the above W_3, W_4 can be made rigorous. We do this as follows:

Theorem 1 *The analytic forms for $W_3(s), W_4(s)$, namely (5), (6) respectively, are finite and well-defined for complex, noninteger s, and for all nonnegative integer s as well. In particular, for nonnegative even integers $s = 2k$, we have*

$$W_3(2k) \;=\; \frac{4^k\Gamma(k+1/2)}{\sqrt{\pi}k!}\,{}_3F_2(-k,-k,-k;1,1/2-k;1/4),$$

$$W_4(2k) \;=\; \frac{4^k\Gamma(k+1/2)}{\sqrt{\pi}k!}\,{}_4F_3(1/2,-k,-k,-k;1,1,1/2-k;1),$$

each of which is a terminating series of all rational terms. For nonnegative odd integers s, we obtain finite limiting forms involving parametric-hypergeometric derivatives.

Proof: All but the last sentence of this theorem follows from the well-known analytic properties of Γ (e.g., poles only at nonpositive integers). As for nonnegative odd s, we employ the useful hypergeometric up-ladder identity

$$ {}_qF_p(a_1,\ldots,a_q;b_1,\ldots,b_p;z) \;=\; 1+z\,\frac{a_1\cdots a_q}{b_1\cdots b_p}\,{}_{q+1}F_{p+1}(1,a_1+1,\ldots,a_q+1;2,b_1+1,\ldots,b_p+1;z), $$

upon which one can recurse until none of the denominator terms in the far-right hypergeometric is a nonpositive integer. Also we use a down-ladder identity

$$ {}_{q+1}F_{p+1}(A,a_1,\ldots,a_q;A,b_1,\ldots,b_p;z) \;=\; {}_qF_p(a_1,\ldots,a_q;b_1,\ldots,b_p;z), $$

all of this together with the known asymptotic

$$ \frac{1}{\Gamma(z)} \;\sim_{z\to 0}\; z+O(z^2), $$

to work out correct limits. In (5) for example, the first hypergeometric term has a numerator Γ divergence at $s=1$, where as the second term can be written

$$ {}_3F_2\left(-\frac{s}{2},-\frac{s}{2},-\frac{s}{2};1,\frac{1}{2}-\frac{s}{2};\frac{1}{4}\right) \;=\; {}_4F_3(1,1-s/2,1-s/2,1-s/2;2,2,3/2-s/2;1/4), $$

whence the ${}_4F_3$ here agrees—at $s=1$—with the *first* ${}_3F_2$ in (5), because of the down-ladder identity. This vanishing of the difference of two hypergeometrics cancels the Γ pole. The same reasoning resolves the W_4 representation (6) at positive odd integers s. (See after this proof some exemplary manipulations along such lines.)

QED

An example of parametric-derivative evaluation is the following where we have taken $s\to 1$ per the limit algorithm implicit in the proof of Theorem 1:

$$ \begin{aligned} W_3(1) \;=\;& (-6+3\gamma+8\log 2)\frac{72\pi^2\Gamma\left(\frac{1}{3}\right)^2-\Gamma\left(-\frac{1}{6}\right)^4\Gamma\left(\frac{2}{3}\right)^2}{4\sqrt{3}\pi^4\Gamma\left(-\frac{1}{6}\right)^2}\;+ \\ & \frac{4}{\pi}\,\partial\,{}_3\tilde{F}_2^{(\{0,0,0\},\{0,1\},0)}\left(\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\{1,0\},\frac{1}{4}\right)\;- \\ & \frac{1}{4\pi}\,\partial\,{}_3\tilde{F}_2^{(\{0,0,0\},\{0,1\},0)}\left(\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\{2,2\},\frac{1}{4}\right)\;+ \\ & \frac{12}{\pi}\,\partial\,{}_3\tilde{F}_2^{(\{0,0,1\},\{0,0\},0)}\left(\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\},\{1,0\},\frac{1}{4}\right), \end{aligned} $$

where the symbology $\partial\,{}_q\tilde{F}_p$ means the derivative of the standard regularized hypergeometric, but with respect to the parameter indicated in the superscript list.

Currently, it is difficult to understand how this limit evaluation of $W_3(1)$ agrees with the spectacular closed form obtained in [3], namely

$$W_3(1) \;=\; \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6(1/3) \;+\; \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6(2/3).$$

(Note that this discovery of those researchers actually evaluates a Meijer-G function at specific parameters; see next section.) The only connection of which the present author is aware is this: Both the derivative evaluation and the Borwein et al. closed form here make use of the third singular value for elliptic integrals—which is how the Γ terms appear in each.

To give the example of limiting procedures for $W_4(1)$, let us denote for simplicity

$$F := {}_4F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2};2,2,2;1\right),$$

$$H := {}_4F_3\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2};1,1,0;1\right).$$

Then the liniting algorithm of Theorem 1 yields

$$W_4(1) \;=\; \frac{1}{2\pi}\left(\psi^{(0)}\left(\frac{3}{2}\right)+2\log 2+\gamma-2\right)F+$$

$$\frac{4}{\pi}\partial\;\tilde{H}^{(\{0,0,0,0\},\{0,0,1\},0)}-\frac{3}{4\pi}\partial\;\tilde{F}^{(\{0,0,0,0\},\{0,0,1\},0)}-$$

$$\frac{1}{\pi}\partial\;\tilde{F}^{(\{0,0,0,1\},\{0,0,0\},0)}+\frac{12}{\pi}\partial\;\tilde{H}^{(\{0,0,0,1\},\{0,0,0\},0)}.$$

So we have cast $W_3(1), W_4(1)$ into finite hypergeometric form, albeit with parametric derivatives involved. For higher odd-integer arguments to W_3, W_4, the procedure is the same: No new function types are involved besides *psi*-functions and hyperderivatives, but the list of terms gets longer and longer as odd integer s gets larger.

5 The W_n, the Meijer-G function, and future research

To continue such research we need a better understanding of Bessel-moment integrals, and of hypergeometrics ${}_pF_q$, especially when $p = q + 1$. The author notes that there is a connection with the mighty Meijer G-functions, and also that integral representations of ${}_{q+1}F_q$ are not too hard to achieve. Such machinations should lead to rigorous proofs of the various Borwein et al. conjectures on the random-walk integrals [3].

One thing that came to the present author as a surprise: If one presses Bessel-moment symbolics somewhat further, one obtains two fascinating analytic representations equivalent to their counterparts in Theorem 1, namely

$$W_3(s) \;=\; -\frac{\Gamma\left(\frac{s}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(-\frac{s}{2}\right)}\; G^{2,1}_{3,3}\left(\frac{1}{4}\;\begin{matrix} 1,1,1 \\ \frac{1}{2},-\frac{s}{2},-\frac{s}{2}\end{matrix}\right),$$

and

$$W_4(s) \;=\; \frac{2^s\,\Gamma\left(\frac{s}{2}+1\right)}{\pi\,\Gamma\left(-\frac{s}{2}\right)}\; G^{2,2}_{4,4}\left(1\,\middle|\,\begin{matrix} 1,\frac{1-s}{2},1,1 \\ \frac{1}{2},-\frac{s}{2},-\frac{s}{2},-\frac{s}{2}\end{matrix}\right),$$

yet we do not know how to generalize, say to W_5 and beyond.[4] Incidentally, the Meijer-G representations here tell us that one approach to calculating W_n is contour integration. In this context, see [1]. It is of interest that explicit values may be calculated rather efficiently, say using contour integration for the Meijer-G values, e.g.:

$$W_3(1) \approx 1.574597237551893657494692183076519690222,$$

$$W_3(3) \approx 6.451679653132074204075792620988788055229,$$

$$W_4(1) \approx 1.799092479842851033532602845846108910066,$$

$$W_4(3) \approx 10.120681010309598571178286893119526237417,$$

all in agreement with the original research of Borwein et al. [3].

There is another fascinating research path, which is to study the “bubble-jump walk,” whereby a particle jumps in D dimensions, always with jump length 1 (and so jumping onto the surface of a bubble whose center is the current position). This should replace the Bessel functions with *elementary* functions for odd D, being as the characteristic spectral function is now elementary itself.[5] What is not clear is how to relate such higher-dimensional jump scenarios to integrals involving amplitudes of complex sums, as is done in definition (1).

6 Acknowledgments

The author is grateful to J. Borwein for pointing out various analytical issues in regard to the integrals at hand. O. Pavlyk lent some monumental expertise in function theory to this project, especially in resolving some Meijer-G forms.

[4]In these stated cases, *Mathematica* will resolve the relevant Meijer-G's for forced integer argument. The present author is indebted to O. Pavlyk for his masterful observations in this context.

[5]Interestingly, the characteristic function of a “dimension-D-bubble” jump involves $J_{D/2-1}$ which is elementary for odd dimension D.

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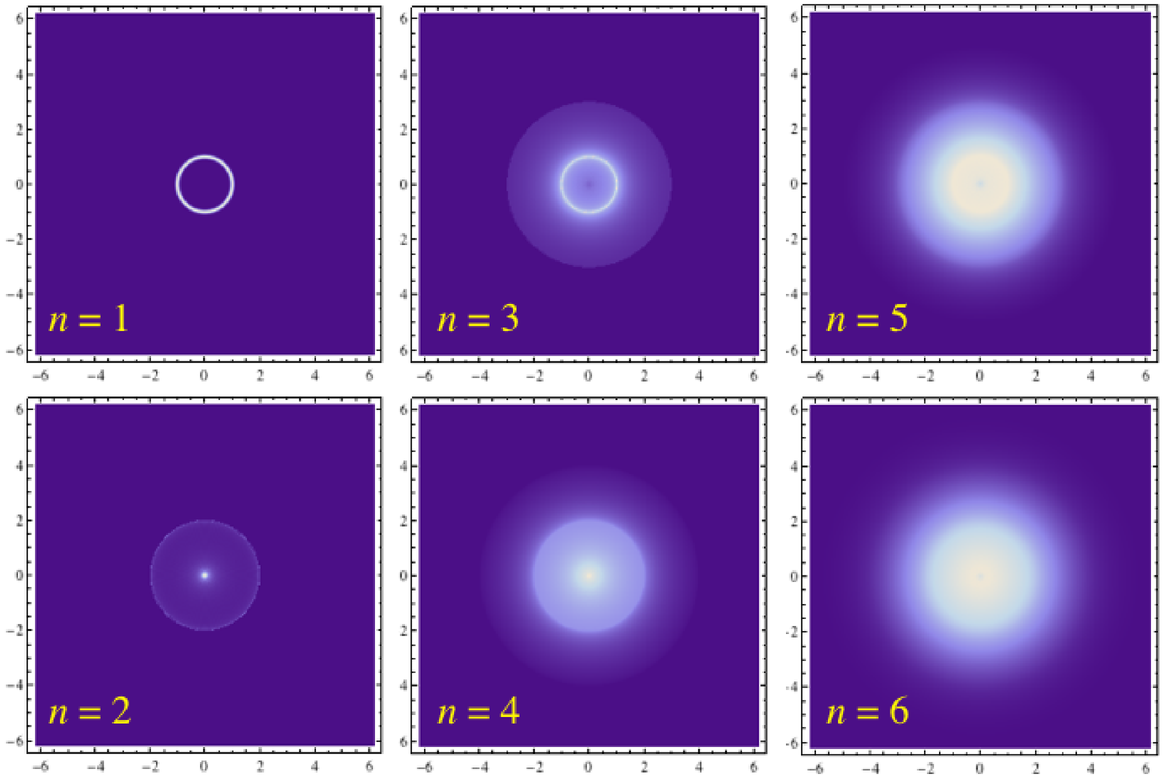

Figure 1: Plots of the spatial densities $\Phi_n(\vec{r}) = \phi_n(r)/(2\pi r)$ for a circle-jump walk after $n = 1, 2, 3, 4, 5, 6$ steps. All plots were obtained via numerical quadrature using integral (3). The $n = 1$ case (upper left) is a delta-circle, as the particle is compelled to jump from origin to radius 1. Looking at the $n = 2$ case (lower left) we see that the particle has an interesting tendency to return to the origin, and also to coalesce at radius $r = 2$. But for $n = 3$ (upper middle), there is a tendency to coalesce at radius $r \sim 1$. Already by $n = 6$ the density is evidently approaching the theoretically demanded large-n Gaussian.

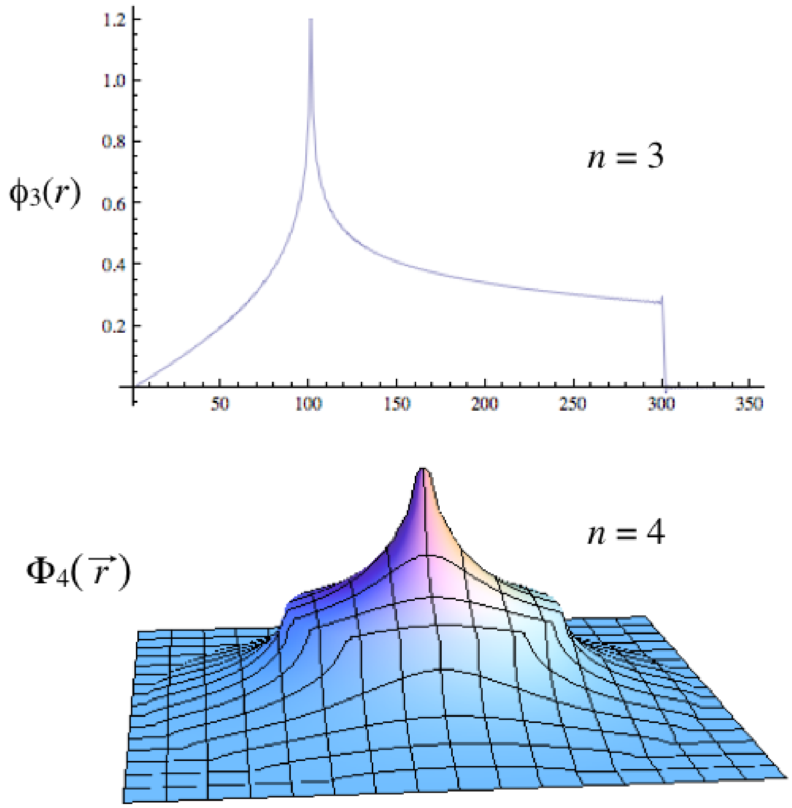

Figure 2: Different views on the probability densities. The upper figure is the *radial* density for $n = 3$, that is, just the function ϕ_3. (The horizontal axis is in units of 0.01 here.) There is an interesting singularity at $r \sim 1$, and, of course, a vanishing for $r > 3$. The lower plot is just another way to view the *spatial* density $\Phi_4(\vec{r})$ (also the lower-middle plot in Figure 1). In spite of such picturesque low-dimensional scenarios, the spatial densities for large n must approach Gaussian forms, as dictated by the classical central-limit theorem.